

A CROSS-INTERSECTION THEOREM FOR VECTOR SPACES BASED ON SEMIDEFINITE PROGRAMMING

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ABSTRACT. Let \mathcal{F} and \mathcal{G} be families of k - and ℓ -dimensional subspaces, respectively, of a given n -dimensional vector space over a finite field \mathbb{F}_q . Suppose that $x \cap y \neq 0$ for all $x \in \mathcal{F}$ and $y \in \mathcal{G}$. By explicitly constructing optimal feasible solutions to a semidefinite programming problem which is akin to Lovász's theta function, we show that $|\mathcal{F}||\mathcal{G}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}$, provided that $n \geq 2k$ and $n \geq 2\ell$. The characterization of the extremal families is also established.

1. INTRODUCTION

Let V be an n -dimensional vector space over a finite field \mathbb{F}_q . Let Ω be the set of subspaces of V . For $k = 0, \dots, n$, let Ω_k be the subset of Ω consisting of the k -dimensional subspaces of V . Two families $\mathcal{F} \subseteq \Omega_k$ and $\mathcal{G} \subseteq \Omega_\ell$ are said to be *cross-intersecting* if $x \cap y \neq 0$ for all $x \in \mathcal{F}$ and $y \in \mathcal{G}$. In the present paper, we apply semidefinite programming to prove the following theorem:

Theorem 1.1. *Let $\mathcal{F} \subseteq \Omega_k$ and $\mathcal{G} \subseteq \Omega_\ell$ be a pair of cross-intersecting families. Suppose that $n \geq 2k$ and $n \geq 2\ell$. Then¹*

$$|\mathcal{F}||\mathcal{G}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix},$$

and equality holds if and only if either (i) there is $z \in \Omega_1$ such that $\mathcal{F} = \{x \in \Omega_k : z \subseteq x\}$ and $\mathcal{G} = \{x \in \Omega_\ell : z \subseteq x\}$, or (ii) $n = 2k = 2\ell$ and there is $z \in \Omega_{2k-1}$ such that $\mathcal{F} = \mathcal{G} = \{x \in \Omega_k : x \subseteq z\}$.

This is a q -analogue of a theorem about cross-intersecting families of subsets of a given n -set, established by Pyber [11] and Matsumoto and Tokushige [9].² Their proofs are based on the Kruskal–Katona theorem (cf. [4]). We may remark that a q -analogue of the Kruskal–Katona theorem has been obtained, but only in a weaker form which is not yet sufficient to prove Theorem 1.1 (cf. [1]).

Linear programming, in particular its duality theory, has been a powerful tool to study various combinatorial configurations. One of the pioneers is Delsarte [2]. His famous LP bound for cliques in association schemes is closely related to Lovász's theta function bound [8] on the Shannon capacities of graphs, but the corresponding SDP problems are drastically reduced to LP by making full use of the regularity of association schemes; cf. [12]. Indeed, it is this method that made it possible to

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¹Here $\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a \\ b \end{bmatrix}_q$ denotes the Gaussian coefficient: $\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=0}^{b-1} ((q^{a-i} - 1)/(q^{b-i} - 1))$.

²The case $k = \ell$ in Theorem 1.1 has recently been settled by Tokushige [17] by using the eigenvalue method.

prove the Erdős–Ko–Rado theorem [3] as well as its q -analogue in full generality; cf. [18, 14, 15]. Theorem 1.1 is an extension of the q -Erdős–Ko–Rado theorem, and we shall consider a ‘bipartite variant’ of an SDP problem defining the theta function. It involves semidefinite constraints of 2×2 matrices even after a reduction, but we are still able to construct optimal feasible solutions to obtain the bound, and then the duality of SDP again provides enough information to describe the extremal families. Recently there have been several attempts to extend Delsarte’s theory by shifting from LP to SDP (cf. [13]),³ but it seems that Theorem 1.1 is among the most successful applications of the duality of SDP to the analysis of optimal combinatorial configurations.

2. A SEMIDEFINITE PROGRAMMING PROBLEM

Let $\mathbb{R}^{\Omega \times \Omega}$ be the set of real matrices with rows and columns indexed by Ω , and let \mathbb{R}^Ω be the set of real column vectors with coordinates indexed by Ω . We shall write $Y \bullet Z = \text{trace}(Y^\top Z)$ for $Y, Z \in \mathbb{R}^{\Omega \times \Omega}$. Let $S\mathbb{R}^{\Omega \times \Omega}$ be the set of symmetric matrices in $\mathbb{R}^{\Omega \times \Omega}$. For $k, \ell = 0, \dots, n$, we define $W_{k,\ell}, \overline{W}_{k,\ell} \in \mathbb{R}^{\Omega \times \Omega}$ by

$$(W_{k,\ell})_{x,y} = \begin{cases} 1 & \text{if } x \in \Omega_k, y \in \Omega_\ell, x \cap y \in \Omega_{\min\{k,\ell\}}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\overline{W}_{k,\ell})_{x,y} = \begin{cases} 1 & \text{if } x \in \Omega_k, y \in \Omega_\ell, x \cap y \in \Omega_0, \\ 0 & \text{otherwise,} \end{cases}$$

for $x, y \in \Omega$. Then $W_{k,\ell} = W_{\ell,k}^\top$ and $\overline{W}_{k,\ell} = \overline{W}_{\ell,k}^\top$. Notice also that $I_k := W_{k,k}$ is the orthogonal projection matrix onto $\mathbb{R}^{\Omega_k} \subseteq \mathbb{R}^\Omega$, and that $\overline{W}_{k,k}$ is (essentially) the adjacency matrix of the q -Kneser graph $qK_{n:k}$. For $k, \ell = 0, \dots, n$, we define $J_{k,\ell} = I_k J I_\ell$, where $J \in \mathbb{R}^{\Omega \times \Omega}$ denotes the all ones matrix.

Let $k, \ell = 1, \dots, \lfloor \frac{n}{2} \rfloor$ and assume for the moment that $k > \ell$.⁴ Let $\mathcal{F} \subseteq \Omega_k$ and $\mathcal{G} \subseteq \Omega_\ell$ be a pair of (nonempty) cross-intersecting families. Let $\varphi, \psi \in \mathbb{R}^\Omega$ be the characteristic vectors of \mathcal{F} and \mathcal{G} , respectively. Then it follows that the matrix

$$X_{\mathcal{F},\mathcal{G}} = \left(\frac{\varphi}{\|\varphi\|} + \frac{\psi}{\|\psi\|} \right) \left(\frac{\varphi}{\|\varphi\|} + \frac{\psi}{\|\psi\|} \right)^\top \in S\mathbb{R}^{\Omega \times \Omega}$$

is a feasible solution to the following SDP problem with objective value $|\mathcal{F}|^{\frac{1}{2}} |\mathcal{G}|^{\frac{1}{2}}$:

$$\begin{aligned} \text{(P):} \quad & \text{maximize} \quad \frac{1}{2} (J_{k,\ell} + J_{\ell,k}) \bullet X \\ & \text{subject to} \quad I_k \bullet X = I_\ell \bullet X = 1, \\ & \quad \quad \quad (\overline{W}_{k,\ell} + \overline{W}_{\ell,k}) \bullet X = 0, \\ & \quad \quad \quad X \succcurlyeq 0, \quad X \geq 0, \end{aligned}$$

³Delsarte’s LP bound is defined on a commutative semisimple \mathbb{C} -algebra associated with an association scheme, called the *Bose–Mesner algebra*. In the process of shifting from LP to SDP, we are also shifting to noncommutative semisimple \mathbb{C} -algebras, such as the *Terwilliger algebra* [16] or a *coherent algebra* [7].

⁴It should be remarked that the assumption $k > \ell$ is just for notational convenience. When $k = \ell$, we understand that Ω_ℓ is a distinct copy of Ω_k , and work instead with matrices with rows and columns indexed by $\Omega \cup \Omega_\ell$.

where $X \in S\mathbb{R}^{\Omega \times \Omega}$ is the variable, and $X \succcurlyeq 0$ (resp. $X \geq 0$) means that X is positive semidefinite (resp. nonnegative). The dual problem is then given by

$$\begin{aligned} \text{(D):} \quad & \text{minimize} \quad \alpha + \beta \\ & \text{subject to} \quad S := \alpha I_k + \beta I_\ell - \frac{1}{2}(J_{k,\ell} + J_{\ell,k}) \\ & \quad \quad \quad -\gamma(\overline{W}_{k,\ell} + \overline{W}_{\ell,k}) - A \succcurlyeq 0, \\ & \quad \quad \quad A \geq 0, \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $A \in S\mathbb{R}^{\Omega \times \Omega}$ are the variables.⁵ Indeed, for any feasible solutions to (P) and (D), we have

$$\begin{aligned} \alpha + \beta - \frac{1}{2}(J_{k,\ell} + J_{\ell,k}) \bullet X &= \left(\alpha I_k + \beta I_\ell - \frac{1}{2}(J_{k,\ell} + J_{\ell,k}) \right) \bullet X \\ &\geq (\gamma(\overline{W}_{k,\ell} + \overline{W}_{\ell,k}) + A) \bullet X \\ &= A \bullet X \\ &\geq 0. \end{aligned}$$

In particular, $(\alpha + \beta)^2$ gives an upper bound on $|\mathcal{F}||\mathcal{G}|$. Furthermore, notice that if $\alpha + \beta = \frac{1}{2}(J_{k,\ell} + J_{\ell,k}) \bullet X$ then $S \bullet X = A \bullet X = 0$. For the rest of this paper, we shall consider the following one-parameter family:

$$(1) \quad \alpha = \beta = \frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}}, \quad \gamma = b(\lambda), \quad A = a(\lambda) \overline{W}_{k,k} + \lambda \overline{W}_{\ell,\ell},$$

where $\lambda \in \mathbb{R}$, and

$$\begin{aligned} q^{k^2}(q^k - 1) \begin{bmatrix} n-k \\ k \end{bmatrix} a(\lambda) &= \frac{1}{2} q^\ell (q^{k-\ell} - 1) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} \\ &\quad + q^{\ell^2} (q^\ell - 1) \begin{bmatrix} n-\ell \\ \ell \end{bmatrix} \lambda, \\ q^{k\ell} \begin{bmatrix} n-k \\ \ell \end{bmatrix} b(\lambda) &= -\frac{1}{2} q^\ell \begin{bmatrix} n-1 \\ \ell \end{bmatrix} - q^{\ell^2} \begin{bmatrix} n-\ell \\ \ell \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{-\frac{1}{2}} \lambda. \end{aligned}$$

In Section 4, we shall prove the following theorem:

Theorem 2.1. *For sufficiently small $\lambda > 0$, (1) gives a feasible solution to (D).*

Theorem 1.1 quickly follows from Theorem 2.1, together with the above comments:

Proof of Theorem 1.1. It follows that $|\mathcal{F}||\mathcal{G}| \leq (\alpha + \beta)^2 = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}$. If equality holds, then $0 = A \bullet X_{\mathcal{F},\mathcal{G}} = a(\lambda) \overline{W}_{k,k} \bullet X_{\mathcal{F},\mathcal{G}} + \lambda \overline{W}_{\ell,\ell} \bullet X_{\mathcal{F},\mathcal{G}}$. However, since $a(\lambda) > 0$ whenever $\lambda > 0$, it follows that $\overline{W}_{k,k} \bullet X_{\mathcal{F},\mathcal{G}} = \overline{W}_{\ell,\ell} \bullet X_{\mathcal{F},\mathcal{G}} = 0$ in this case. In other words, each of \mathcal{F} and \mathcal{G} is an intersecting family. From the q -Erdős–Ko–Rado theorem (cf. [10, 6, 14]) it follows that $|\mathcal{F}| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ and $|\mathcal{G}| = \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}$, and the description of \mathcal{F} and \mathcal{G} also easily follows from that theorem. \square

Remark 2.2. It is possible to consider SDP problems likewise for cross t -intersecting families (that is to say, we require that $\dim(x \cap y) \geq t$ for all $x \in \mathcal{F}$ and $y \in \mathcal{G}$). However, constructing appropriate optimal feasible solutions for $t \geq 2$ appears to be a quite complicated problem, except when $k = \ell$, in which case it turns out

⁵From the constraints it follows that $A_{x,y} = 0$ unless $x, y \in \Omega_k \cup \Omega_\ell$.

that the problem reduces to LP. See also [17]. We shall discuss cross t -intersecting families in future papers.

3. BLOCK DIAGONALIZATION

Frankl and Wilson [5] obtained the bound in the q -Erdős–Ko–Rado theorem. This is an application of Delsarte’s LP bound, and involves (among other results) calculations of the eigenvalues of the q -Kneser graphs. In this section, we shall follow their method to block-diagonalize the matrices defining (P) and (D).

For $i, k, \ell = 0, \dots, n$, it follows that ([5, pp. 231–232])⁶

$$(2) \quad W_{k,\ell} W_{\ell,i} = \begin{bmatrix} k-i \\ \ell-i \end{bmatrix} W_{k,i} \quad \text{if } i \leq \ell \leq k,$$

$$(3) \quad W_{i,k} \overline{W}_{k,\ell} = q^{\ell(k-i)} \begin{bmatrix} n-i-\ell \\ k-i \end{bmatrix} \overline{W}_{i,\ell} \quad \text{if } i \leq k,$$

$$(4) \quad \overline{W}_{k,\ell} = \sum_{h=0}^{\min\{k,\ell\}} (-1)^h q^{\binom{h}{2}} W_{k,h} W_{h,\ell}.$$

We define subspaces U_i ($i = 0, \dots, \lfloor \frac{n}{2} \rfloor$) of \mathbb{R}^Ω as follows:

$$U_i = \{\mathbf{u} \in \mathbb{R}^\Omega : W_{i-1,i} \mathbf{u} = 0\},$$

where $W_{-1,0} := 0$. Then

Lemma 3.1. *For $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$, $k = 0, \dots, n$ and $\mathbf{u} \in U_i$, we have $W_{k,i} \mathbf{u} = 0$ if $k < i$ or $k > n - i$. Moreover,*

$$\overline{W}_{k,i} \mathbf{u} = (-1)^i q^{\binom{i}{2}} W_{k,i} \mathbf{u}.$$

Proof. From (2) it follows that $W_{k,i} \mathbf{u} = 0$ if $k < i$. The other assertions follow from this and (4). \square

The next two lemmas are easy consequences of Lemma 3.1 and (3):

Lemma 3.2. *For $i, j = 0, \dots, \lfloor \frac{n}{2} \rfloor$, $k = 0, \dots, n$, $\mathbf{u} \in U_i$ and $\mathbf{v} \in U_j$, we have*

$$\mathbf{u}^\top W_{i,k} W_{k,j} \mathbf{v} = \delta_{i,j} q^{i(k-i)} \begin{bmatrix} n-2i \\ k-i \end{bmatrix} \mathbf{u}^\top \mathbf{v}.$$

Lemma 3.3. *For $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$, $k, \ell = 0, \dots, n$ and $\mathbf{u} \in U_i$, we have*

$$\overline{W}_{k,\ell} W_{\ell,i} \mathbf{u} = (-1)^i q^{\binom{i}{2} + k(\ell-i)} \begin{bmatrix} n-k-i \\ \ell-i \end{bmatrix} W_{k,i} \mathbf{u}.$$

For $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$, we fix an orthonormal basis $\mathbf{u}_{i,1}, \mathbf{u}_{i,2}, \dots, \mathbf{u}_{i,d_i}$ of U_i , where $d_i := \dim U_i = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix}$. We moreover define

$$\mathbf{u}_{i,r}^k = q^{-\frac{i(k-i)}{2}} \begin{bmatrix} n-2i \\ k-i \end{bmatrix}^{-\frac{1}{2}} W_{k,i} \mathbf{u}_{i,r} \quad (r = 1, \dots, d_i, k = i, \dots, n-i).$$

From Lemmas 3.2 and 3.3 it follows that

⁶For integers a and b , we interpret $\begin{bmatrix} a \\ b \end{bmatrix} = 0$ if $a < 0$ or $b < 0$.

Proposition 3.4. *The $\mathbf{u}_{i,r}^k$ form an orthonormal basis of \mathbb{R}^Ω . Furthermore, for $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$, $\ell = i, \dots, n-i$, $k = 0, \dots, n$ and $r = 1, \dots, d_i$, we have*

$$\overline{W}_{k,\ell} \mathbf{u}_{i,r}^\ell = \theta_i^{k,\ell} \mathbf{u}_{i,r}^k,$$

where $\mathbf{u}_{i,r}^k := 0$ if $k < i$ or $k > n-i$, and

$$\theta_i^{k,\ell} = (-1)^i q^{\binom{i}{2} + k\ell - \frac{i(k+\ell)}{2}} \begin{bmatrix} n-k-i \\ \ell-i \end{bmatrix} \begin{bmatrix} n-2i \\ k-i \end{bmatrix} \begin{bmatrix} n-2i \\ \ell-i \end{bmatrix}^{-\frac{1}{2}}.$$

Notice that $\theta_i^{k,\ell} = 0$ if $k < i$ or $k > n-i$, and that $\theta_i^{k,\ell} = \theta_i^{\ell,k}$ for $k, \ell = i, \dots, n-i$. We also need the following observation:

Lemma 3.5. *For $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$, $\ell = i, \dots, n-i$, $k = 0, \dots, n$ and $r = 1, \dots, d_i$, we have*

$$J_{k,\ell} \mathbf{u}_{i,r}^\ell = \delta_{i,0} \begin{bmatrix} n \\ \ell \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^{\frac{1}{2}} \mathbf{u}_{0,1}^k.$$

4. PROOF OF THEOREM 2.1

In this section, we shall prove Theorem 2.1. Let $k, \ell = 1, \dots, \lfloor \frac{n}{2} \rfloor$, and recall that we are assuming that $k > \ell$.⁷ Let $\lambda > 0$. Then $a(\lambda) > 0$, so that (1) gives a feasible solution to (D) if and only if the matrix

$$S(\lambda) = \frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} (I_k + I_\ell) - \frac{1}{2} (J_{k,\ell} + J_{\ell,k}) \\ - a(\lambda) \overline{W}_{k,k} - \lambda \overline{W}_{\ell,\ell} - b(\lambda) (\overline{W}_{k,\ell} + \overline{W}_{\ell,k})$$

is positive semidefinite. By virtue of Proposition 3.4 and Lemma 3.5, $S(\lambda) \succcurlyeq 0$ if and only if

$$S_0(\lambda) = \begin{pmatrix} \frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} - \theta_0^{k,k} a(\lambda) & -\frac{1}{2} \begin{bmatrix} n \\ k \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}^{\frac{1}{2}} - \theta_0^{k,\ell} b(\lambda) \\ -\frac{1}{2} \begin{bmatrix} n \\ k \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}^{\frac{1}{2}} - \theta_0^{k,\ell} b(\lambda) & \frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} - \theta_0^{\ell,\ell} \lambda \end{pmatrix} \succcurlyeq 0,$$

$$S_i(\lambda) = \begin{pmatrix} \frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} - \theta_i^{k,k} a(\lambda) & -\theta_i^{k,\ell} b(\lambda) \\ -\theta_i^{k,\ell} b(\lambda) & \frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} - \theta_i^{\ell,\ell} \lambda \end{pmatrix} \succcurlyeq 0,$$

for $i = 1, \dots, \ell$, and

$$s_i(\lambda) = \frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} - \theta_i^{k,k} a(\lambda) \geq 0,$$

for $i = \ell + 1, \dots, k$.

⁷See footnote 4. The discussions in this section work for the case $k = \ell$ as well, and in fact become much simpler.

For $i = 0, \dots, k$, we now calculate:

$$\begin{aligned}
 (5) \quad \left| \theta_i^{k,k} a(0) \right| &= \frac{1}{2} q^{\binom{i}{2} + \ell - ik} \frac{q^{k-\ell} - 1}{q^k - 1} \begin{bmatrix} n-k-i \\ k-i \end{bmatrix} \begin{bmatrix} n-k \\ k \end{bmatrix}^{-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} \\
 &\leq \frac{1}{2} q^{\ell - \frac{ik}{2}} \frac{q^{k-\ell} - 1}{q^k - 1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} \\
 &< \begin{cases} \frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} & \text{for } i = 0, 1, \\ \frac{1}{2q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} & \text{for } i = 2, \dots, k. \end{cases}
 \end{aligned}$$

Therefore the diagonal entries of the $S_i(\lambda)$ ($i = 0, \dots, \ell$) as well as the $s_i(\lambda)$ ($i = \ell + 1, \dots, k$) are positive, provided that λ is sufficiently small. Then it is readily verified that $S_0(\lambda)$ and $S_1(\lambda)$ are scalar multiples of the rank 1 matrices⁸

$$\begin{pmatrix} q^\ell - 1 & -(q^k - 1)^{\frac{1}{2}}(q^\ell - 1)^{\frac{1}{2}} \\ -(q^k - 1)^{\frac{1}{2}}(q^\ell - 1)^{\frac{1}{2}} & q^k - 1 \end{pmatrix}$$

and

$$\begin{pmatrix} q^\ell(q^{n-\ell} - 1) & -q^{\frac{k+\ell}{2}}(q^{n-k} - 1)^{\frac{1}{2}}(q^{n-\ell} - 1)^{\frac{1}{2}} \\ -q^{\frac{k+\ell}{2}}(q^{n-k} - 1)^{\frac{1}{2}}(q^{n-\ell} - 1)^{\frac{1}{2}} & q^k(q^{n-k} - 1) \end{pmatrix},$$

respectively, from which it follows that $S_0(\lambda) \succ 0$ and $S_1(\lambda) \succ 0$. Hence it is enough to show that $\det(S_i(0)) > 0$ for $i = 2, \dots, \ell$, i.e.,

$$\frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} \left(\frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} - \theta_i^{k,k} a(0) \right) > \left(\theta_i^{k,\ell} b(0) \right)^2$$

for $i = 2, \dots, \ell$. On one hand, it follows from (5) that the left-hand side is strictly larger than $\frac{1}{4}(1 - \frac{1}{q}) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}$. On the other hand, since

$$\left(\frac{\theta_{i+1}^{k,\ell}}{\theta_i^{k,\ell}} \right)^2 = \frac{q^{2i-k-\ell}(q^{k-i} - 1)(q^{\ell-i} - 1)}{(q^{n-k-i} - 1)(q^{n-\ell-i} - 1)} < 1 \quad (i = 2, \dots, \ell - 1),$$

it follows that

$$\begin{aligned}
 \left(\theta_i^{k,\ell} b(0) \right)^2 &\leq \left(\theta_2^{k,\ell} b(0) \right)^2 \\
 &= \frac{1}{4} q^{2-2k} \frac{q^{k-1} - 1}{q^{n-k-1} - 1} \frac{q^{\ell-1} - 1}{q^{n-\ell-1} - 1} \frac{q^{n-\ell} - 1}{q^{n-k} - 1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix} \\
 &\leq \frac{1}{4} q^{2-2k} \frac{q^{n-\ell} - 1}{q^{n-k} - 1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix} \\
 &< \frac{1}{4} q^{(2-2k)+(k-\ell+1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix} \\
 &\leq \frac{1}{4q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}
 \end{aligned}$$

⁸In fact, we arrived at this condition conversely with the help of the duality, i.e., that $S \bullet X_{\mathcal{F}, \mathcal{G}} = 0$ for ‘extremal’ cross-intersecting families \mathcal{F} and \mathcal{G} given in Theorem 1.1. The constraints of (D) and the condition together led us to the one-parameter family (1).

for $i = 2, \dots, \ell$. Hence $\det(S_i(0)) > 0$ for $i = 2, \dots, \ell$, and the proof is complete.

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